Chapter 5

Lecture 4

Rutherford Scattering

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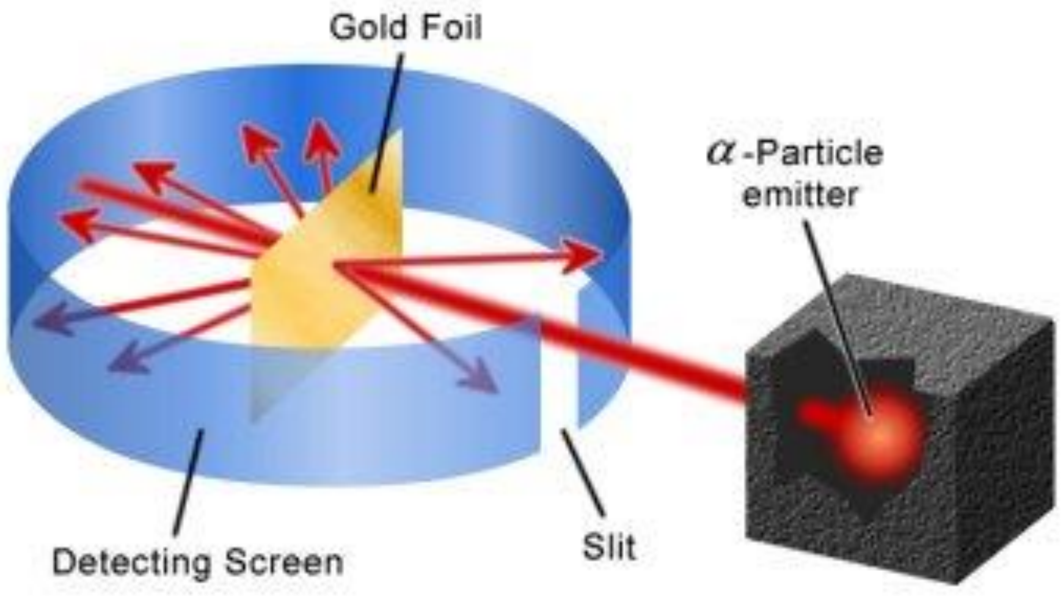
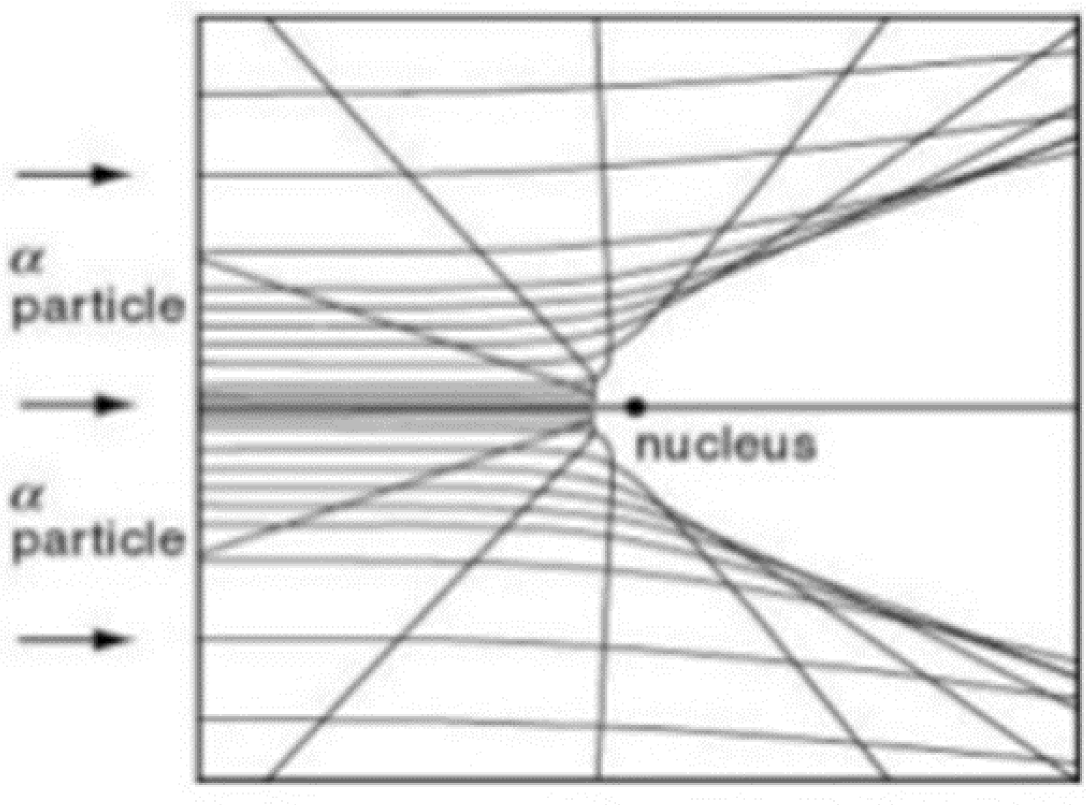
5.5 Rutherford Scattering Formula

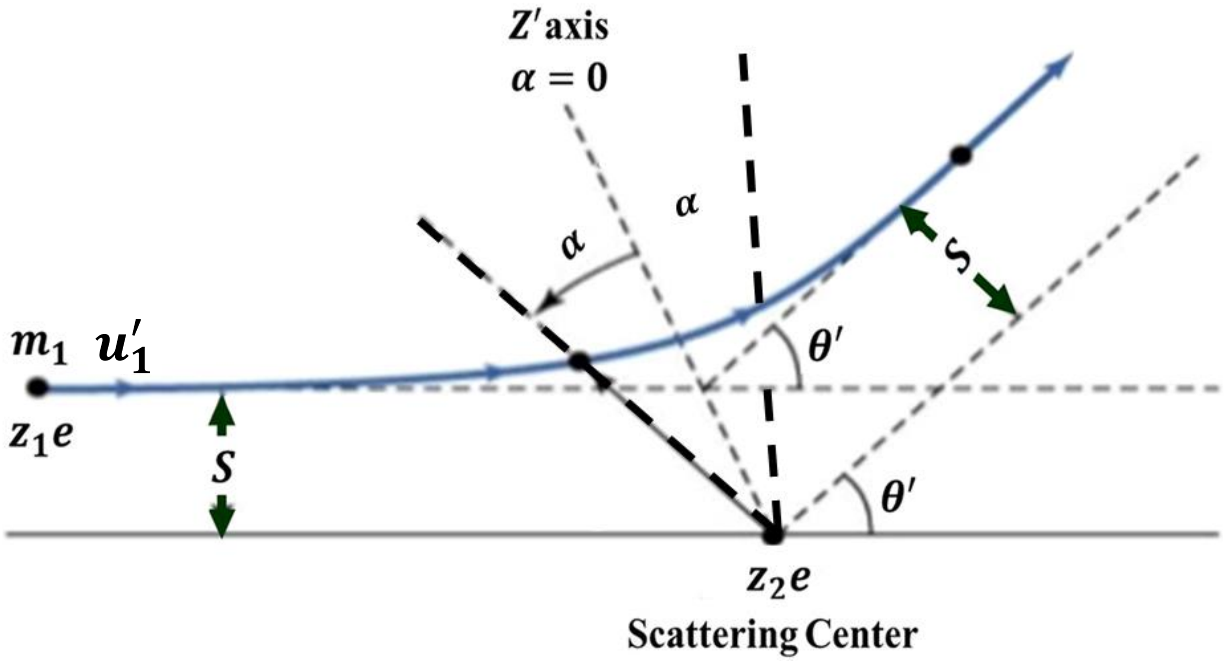
Rutherford's experiment of α-particles scattering by the atoms in a thin foil of gold revealed the existence of positively charged nucleus in the atom.

- The α-particles were scattered in all directions.
- Some passes undeflected by the foil.
- Some were scattered through larger angles even back scattered.
- The only valid explanation of large angle deflection was if the total positive charge were concentrated at some small region.
- Rutherford called it nucleus.



5.5 Rutherford Scattering Formula





5.5 Rutherford Scattering Formula

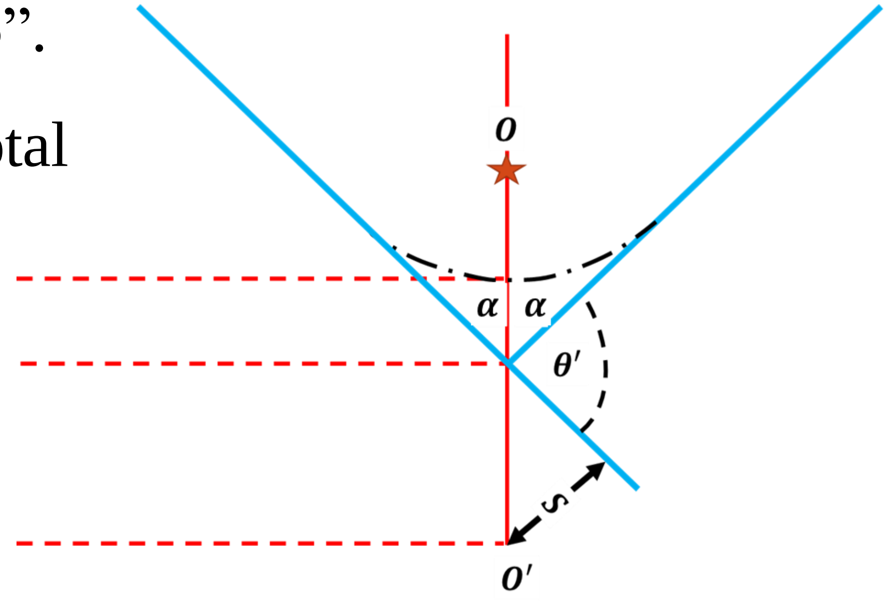
Consider an incident charge particle with charge z_1e is scattered through a target with charge z_2e.

The trajectory of scattered particles is a hyperbola as shown.

The trajectory of the particles in repulsive field with origin at o' is shown.

In the case of attractive force, center will be at "o".

when particles are apart at large distance, the total energy it posses is in form of Kinetic energy



$$k' = {}^{1}\!/_{2}\ \mu {u_1'}^2 = E$$

$$u_1' = \sqrt{\frac{2E}{\mu}}$$

5.5 Rutherford Scattering Formula

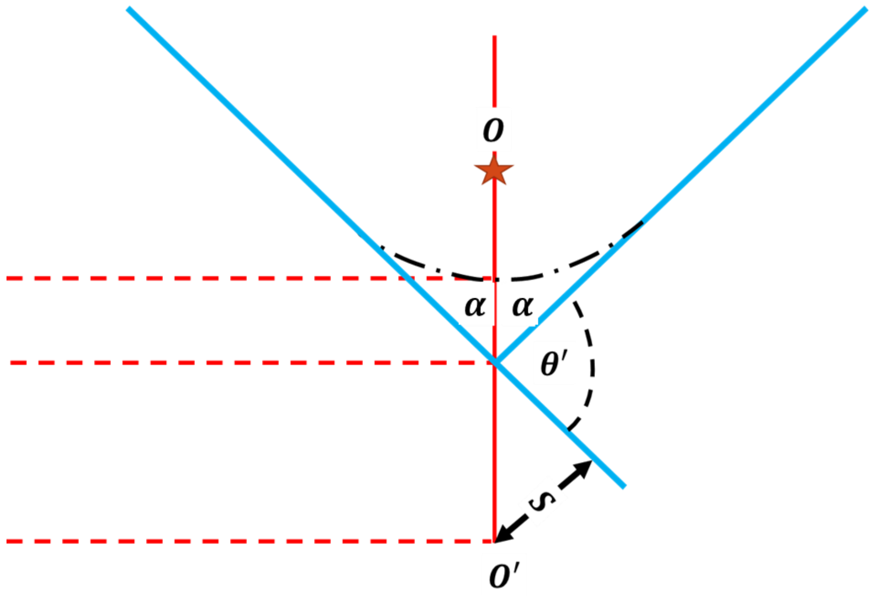
We have to find $\sigma(\theta') = \frac{s}{\sin\theta'}\left|\frac{ds}{d\theta'}\right|$

$$l = \mu u_1' S = S\sqrt{2\mu E}$$

Where S is the impact parameter. The trajectory is hyperbola, from equation of conic

$$\frac{\alpha}{r} = e\cos\alpha - 1 \quad \text{and} \qquad \text{for } r \to \infty$$

$$\Rightarrow \cos\alpha = \frac{1}{e} = \frac{1}{\sqrt{1+\frac{2El^2}{\mu k^2}}}$$



From the figure $$2\alpha + \theta' = \pi$$

$$\alpha = \frac{\pi - \theta'}{2}$$

Now $$\cot\alpha = \cot\left(\frac{\pi-\theta'}{2}\right) = \tan\frac{\theta'}{2}$$

5.5 Rutherford Scattering Formula

$$\cot\alpha = \tan\frac{\theta'}{2}$$

$$\tan\frac{\theta'}{2} = \frac{\cos\alpha}{\sin\alpha} = \frac{\cos\alpha}{\sqrt{1-cos^2\alpha}}$$

$$\tan\frac{\theta'}{2} = \frac{\cos\alpha}{\cos\alpha\sqrt{\frac{1}{cos^2\alpha}-1}}$$

$$\tan\frac{\theta'}{2} = \frac{1}{\sqrt{\frac{1}{cos^2\alpha}-1}}$$

$$\tan\frac{\theta'}{2} = \frac{1}{\sqrt{e^2-1}} = \frac{1}{\sqrt{1+\frac{2El^2}{\mu k^2}-1}}$$

5.5 Rutherford Scattering Formula

$$\tan\frac{\theta'}{2} = \left(\frac{2El^2}{\mu k^2}\right)^{-1/2}$$

$$\tan\frac{\theta'}{2} = \frac{k}{l}\sqrt{\frac{\mu}{2E}} = \frac{k}{lu_1'}$$

Because $u_1' = \sqrt{\frac{2E}{\mu}}$

Since $l = \mu u_1' S$

$$\boxed{\Rightarrow \tan\frac{\theta'}{2} = \frac{k}{S\mu u_1'^2}}$$

Differentiating with θ' above equation can be written as.

$$\frac{1}{2}sec^2\frac{\theta'}{2} = \frac{-k}{S^2\mu u_1'^2}\frac{ds}{d\theta'}$$

$$\Rightarrow \left|\frac{ds}{d\theta'}\right| = \frac{S^2\mu u_1'^2}{2k\,cos^2\frac{\theta'}{2}}$$

5.5 Rutherford Scattering Formula

Since the differential scattering Cross-Section is given by.

$$\sigma(\theta') = \frac{s}{\sin\theta'}\left|\frac{ds}{d\theta'}\right|$$

$$\sigma(\theta') = \frac{s}{2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}}\frac{s^2\mu u_1'^2}{2k\cos^2\frac{\theta'}{2}}$$

$$\boxed{\sigma(\theta') = \frac{s^3\mu u_1'^2}{4k\sin\frac{\theta'}{2}\cos^3\frac{\theta'}{2}}}$$

Since $\tan\frac{\theta'}{\mathbf{2}} = \frac{k}{s\mu u_1'^2}$

$$\Rightarrow s = \frac{k}{\mu u_1'^2}\frac{\cos\frac{\theta'}{\mathbf{2}}}{\sin\frac{\theta'}{\mathbf{2}}}$$ Putting in above equation

5.5 Rutherford Scattering Formula

$$\sigma(\theta') = \left(\frac{k^3}{\mu^3 {u_1'}^6} \frac{cos^3\frac{\theta'}{2}}{sin^3\frac{\theta'}{2}} \right) \cdot \frac{\mu {u_1'}^2}{4k\ sin\frac{\theta'}{2} cos^3\frac{\theta'}{2}}$$

$$\sigma(\theta') = \frac{k^2}{\mu^2 {u_1'}^4} \cdot \frac{1}{4\ sin^4\frac{\theta'}{2}}$$

$$\sigma(\theta') = \frac{k^2}{4E^2} \cdot \frac{1}{4sin^4\frac{\theta'}{2}}$$

Since $|k| = z_1 z_2 e^2$, Therefore,

$$\boxed{\sigma(\theta') = \frac{k^2}{4E^2} \cdot \frac{1}{4sin^4\frac{\theta'}{2}} = \frac{(z_1 z_2 e^2)^2}{4E^2} \cdot \frac{1}{4sin^4\frac{\theta'}{2}}}$$

5.5 Rutherford Scattering Formula

The total cross section in Centre of mass coordinate system will be

$$\sigma_{C.M}(\theta') = 2\pi \int \sigma(\theta') \sin\theta' \, d\theta'$$

$$\sigma_{C.M}(\theta') = 2\pi \int \frac{\left(z_1 z_2 e^2\right)^2}{4E^2} . \frac{1}{4sin^4\frac{\theta'}{2}} \sin\theta' \, d\theta'$$

$$\sigma_{C.M}(\theta') = \pi \frac{\left(z_1 z_2 e^2\right)^2}{2E^2} \int \frac{1}{4sin^4\frac{\theta'}{2}} \left(2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}\right) d\theta'$$

$$\sigma_{C.M}(\theta') = \frac{z_1^2 z_2^2 e^4 \pi}{4E^2} \int cosec^3\frac{\theta'}{2} \;\; \cos\frac{\theta'}{2} \, d\theta'$$

5.5 Rutherford Scattering Formula

If $z_1 = 2, z_2 = 79$, e $= 1.6 \times 10^{-19} C, E = 7.9 MeV$ & $r_0 = 10^{-14}$m.

- If we solve the question for $\boldsymbol{\sigma_T(\theta')}$ we get a divergent results.
- The physical reason is that coulomb field, which has infinite range.
- Particles with large impact parameters will be deflected through some angle.
- Hence the small but finite contribution led us to an infinite value for the $\boldsymbol{\sigma_{C.M}(\theta')}$.
- In case of atoms, the nuclear coulomb field is screened by electrons around the nucleus and results in finite range and finite cross-section.

Such field is represented by

$$V = \frac{1}{r} e^{-r/a}$$

"a is the screening radius.

5.5 Rutherford Scattering Formula

Distance of Closest approach

The distance at which scattered particles turned away from the scattering center.

$$r_1 = a(1 + e) \quad \text{and} \quad a = \frac{k}{2E} = \frac{z_1 z_2 e^2}{2E}$$

$$r_1 = \frac{z_1 z_2 e^2}{2E}\left[1 + \sqrt{1 + \frac{2El^2}{z_1^2 z_2^2 e^4 \mu}}\right]$$

For the smallest distance l must be zero.

Therefore,

$$r_{1(min)} = r_0 = \frac{z_1 z_2 e^2}{2E}[1 + 1]$$

$$r_{1(min)} = \frac{z_1 z_2 e^2}{E}$$

$$E = \frac{z_1 z_2 e^2}{r_0}$$

5.6 Examine the scattering Produced by a Repulsive central force $f = kr^{-3}$

$\sigma(\theta)d\theta = \frac{k}{2E} \cdot \frac{(1-x)dx}{x^2(2-x)^2 \sin\pi x}$, where x is the ratio ${}^{\theta_s}/_{\pi}$ and E is the energy.

Solution: The repulsive central force is $f = {}^{k}/_{r^3} = ku^3$ and the differential equation for the orbit is

$$\frac{d^2u}{d\theta^2} + u = -\frac{\mu f_{(u)}}{l^2u^2} = -\frac{\mu k u^3}{l^2u^2}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + \left(1 + \frac{\mu k}{l^2}\right)u = 0$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + \gamma^2 u = 0$$

$$\gamma = \sqrt{1 + \frac{\mu k}{l^2}} \qquad (1)$$

The solution of this differential equation is

$$u = A\cos\gamma\theta + \mathrm{B}\sin\gamma\theta$$

Before the collision particles is at angle $\theta = \pi$ and at $r = \infty$

$$u(\theta = \pi) = 0$$

$$\Rightarrow A\cos\gamma\pi + B\cos\gamma\pi = 0$$

$$\Rightarrow A = -B\tan\gamma\pi \qquad \text{(a)}$$

After collision to $r = \infty$ at angle $\theta = \theta_s$ yield the condition

$$A\cos\gamma\theta_s + B\sin\gamma\theta_s = 0$$

Putting the value of A

$$\Rightarrow -B\tan\gamma\pi\cos\gamma\theta_s + B\sin\gamma\theta_s = 0$$

$$\Rightarrow -\tan\gamma\pi\cos\gamma\theta_s + \sin\gamma\theta_s = 0$$

$$\Rightarrow -\frac{\sin\gamma\pi}{\cos\gamma\pi}\cos\gamma\theta_s + \sin\gamma\theta_s = 0$$

$$\Rightarrow -\sin\gamma\pi\cos\gamma\theta_s + \sin\gamma\theta_s\cos\gamma\pi = 0$$

$$\Rightarrow \sin\gamma(\theta_s - \pi) = 0$$

$$\Rightarrow \gamma(\theta_s - \pi) = \sin^{-1}(0) = \pi$$

$$\Rightarrow \gamma = \frac{\pi}{\theta_s - \pi} = \frac{\pi}{\pi\left(\frac{\theta_s}{\pi} - 1\right)}$$

In terms of $x = {}^{\theta}/_{\pi}$

$$\boldsymbol{\gamma = \frac{1}{x-1}}$$

Putting in equation (1) and squaring both the sides

$$1 + \frac{\mu k}{l^2} = \frac{1}{(x-1)^2}$$

$$\Rightarrow 1+\frac{\mu k}{(2\mu E S^2)}=\frac{1}{(x-1)^2}$$

$$\Rightarrow \frac{\mu k}{2\mu E S^2}=\frac{1}{(x-1)^2}-1$$

$$\Rightarrow \frac{k}{2ES^2}=\frac{1-(x-1)^2}{(x-1)^2}$$

$$\Rightarrow \frac{k}{2ES^2}=\frac{1-1-x^2+2x}{(x-1)^2}=\frac{-x(x-2)}{(x-1)^2}$$

$$\Rightarrow S^2=-\frac{k}{2E}\frac{(x-1)^2}{x(x-2)}=-\frac{k}{2E}(x-1)^2x^{-1}(x-2)^{-1}$$

Differentiating above equation

$$\Rightarrow 2SdS=-\frac{k}{2E}\left[\frac{2(x-1)}{x(x-2)}-\frac{(x-1)^2}{x^2(x-2)}-\frac{(x-1)^2}{x(x-2)^2}\right]dx$$

$$\Rightarrow 2SdS = -\frac{k}{2E}\left[\frac{2x(x-1)(x-2)-(x-1)^2(x-2)-x(x-1)^2}{x^2(x-2)^{\ 2}}\right]\mathrm{dx}$$

$$\Rightarrow 2SdS = -\frac{k}{2E}\left[(x-1)\left\{\frac{2x(x-2)-(x-1)(x-2)-x(x-1)}{x^2(x-2)^{\ 2}}\right\}\right]\mathrm{dx}$$

$$\Rightarrow 2SdS = -\frac{k}{2E}\left[(x-1)\left\{\frac{2x^2-4x-x^2+3x-2-x^2+x}{x^2(x-2)^{\ 2}}\right\}\right]\mathrm{dx}$$

$$\Rightarrow 2SdS = -\frac{k}{2E}\left[(1-x)\left\{\frac{-2}{x^2(x-2)^{\ 2}}\right\}\right]\mathrm{dx}$$

$$\Rightarrow SdS = -\frac{k}{2E}\,\frac{(1-x)}{x^2(2-x)^{\ 2}}\,\mathrm{dx}$$

No the differential cross section is given by

$$\sigma(\theta)d\theta = \frac{|SdS|}{sin\ \theta}$$

$$\boxed{\Rightarrow \sigma(\theta)d\theta = \frac{k}{2E}\,\frac{(1-x)}{x^2(2-x)^{\ 2}}\,\frac{1}{sin\ \pi x}\,\mathrm{dx}}$$